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NOTE

THE CIRCULAR FLOW NUMBER OF A 6-EDGE CONNECTED GRAPH IS LESS THAN FOUR

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We show that every 6-edge connected graph admits a circulation whose range lies in the interval $[1,3)$.

The *circular flow number* $\phi_c(G)$ of a (finite) graph G is defined by

$$\phi_c(G) = \inf\{r \in \mathbb{R} : \text{ some orientation } \vec{G} \text{ admits a circulation } f : E(\vec{G}) \to [1, r-1]\}.$$

This parameter is a refinement of the well studied *flow number* $\phi(G) := \lceil \phi_c(G) \rceil$, which was introduced by Tutte as a dual to the chromatic number. Since $\phi_c(G) \geq 2$ with equality if and only if G is eulerian, the circular flow number may be regarded as a measure of how close a graph is to being eulerian. The following results and conjectures can be found in [5].

- **Theorem** (Seymour, 1981) Every 2-edge connected graph G has $\phi_c(G) \leq 6$.
- **Conjecture** (Tutte, 1954) Every 2-edge connected graph G has $\phi_c(G) \leq 5$.

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– **Theorem** (Jaeger, 1975) Every 4-edge connected graph G has $\phi_c(G) \leq 4$.
– **Conjecture** (Tutte, 1966) Every 4-edge connected graph G has $\phi_c(G) \leq 3$.

This list suggests that the circular flow number might decrease as edge connectivity increases. For this and other reasons, the following has been asked [1].

Question 1. Is it true that $\sup\{\phi_c(G) : G \text{ is } k\text{-edge connected}\} \to 2$ as $k \to \infty$?

The answer is "yes" for graphs of bounded genus [6], but little progress has been made for general graphs. In this note we present a refinement of Jaeger's result.

Theorem 2. *Every 6-edge connected graph G has $\phi_c(G) < 4$.*

It suits our purpose to use the following "Minty-like" formula for ϕ_c which arises directly from Hoffman's circulation condition ([3], or see [2]). If $G = (V, E)$ is finite and 2-edge connected, then

$$\phi_c(G) = \min_{\vec{G}} \max_{\emptyset \neq X \subset V} \frac{|\delta X|}{|\delta^+ X|} \tag{1}$$

where $\vec{G}$ ranges over the strong orientations of G. (Here $\delta^+ X$ denotes the set of arcs from X to $V - X$, and $\delta X = \delta^+ X \cup \delta^+(V - X)$.)

Let $T \subseteq V(G)$. A *T-join* in G is a subset $J \subseteq E(G)$ such that T is the set of odd-degree vertices in the induced subgraph $G[J]$. An $\emptyset$-join is usually called a *cycle* or *even subgraph* of G. We use two standard results regarding trees and T-joins. The first is folklore, and the second was first proved by Nash-Williams [4].

Lemma 3. *Any tree H contains a T-join, for any $T \subseteq V(H)$ of even cardinality.*

Lemma 4. *Any $2k$-edge connected graph contains k edge-disjoint spanning trees.*

Lemma 5. *Let H_1 and H_2 be edge-disjoint spanning trees of a graph G and let T be an even subset of $V(G)$. Then $H_1 \cup H_2$ contains a T-join which is spanning and connected.*

Proof. Let V_1 be the set of odd-degree vertices in H_1. The symmetric difference $V_1 \Delta T$ has even cardinality, so, by Lemma 3, H_2 contains a $(V_1 \Delta T)$-join J_2. Let $F = H_1 \cup J_2$. Since H_1 and J_2 are edge disjoint, F is a T-join. Furthermore $E(H_1) \subseteq F$ so F spans G and is connected. ∎

Lemma 6. *Consider the polyhedron* $P = \{x \in \mathbb{R}^8 : Ax = b,\, x \geq 0\}$ *where*

$$[A|b] = \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

Then the linear program $z^* = \min\{[1\,1\,1\,1\,0\,0\,0\,0]x : x \in P\}$ *has a unique optimum solution* $x^* = \left[\frac{1}{4}\,0\,0\,0\,0\,0\,\frac{1}{4}\,\frac{1}{2}\right]^T$ *with value* $z^* = \frac{1}{4}$.

Proof. It is routine to check that x^* is P-feasible and that the vector $y^* = \left[\frac{1}{2}\,\frac{1}{4}\,\frac{1}{4}\right]$ is a feasible solution to the dual linear program $\max\{y[0\,0\,1]^T : yA \leq [1\,1\,1\,1\,0\,0\,0\,0]\}$. Both objective values equal $\frac{1}{4}$ so (x^*, y^*) is an optimal dual pair. To show uniqueness we demonstrate that the primal objective vector is in the strict interior of a full dimensional cone generated by normals of active (tight) constraints at x^*. Writing $x = [x_1\, x_2\, \dots\, x_8]^T$, the active constraints are the three equations $Ax = b$ and the five equations $x_i = 0$, $i = 2,3,4,5,6$. Indeed we have the positive linear combination

$$[1\,1\,1\,1\,0\,0\,0\,0] = \left[\frac{1}{2}\;\;\frac{1}{4}\;\;\frac{1}{4}\right]A + \frac{1}{4}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{4}e_6$$

where e_i is the i-th standard unit vector in $\mathbb{R}^8$. The cone is full dimensional since the first, seventh and eighth columns of A are linearly independent. ∎

Proof of Theorem 2. Let V_1 be the vertices of odd degree in G. By Lemma 4, G has three edge disjoint spanning trees. So by Lemmas 3 and 5, G has two edge-disjoint V_1-joins, J_1, J_2, such that J_2 spans G and is connected. Let $\vec{C_1}$ and $\vec{C_2}$ be eulerian orientations of the complementary cycles $C_1 = E - J_1$ and $C_2 = E - J_2$. Note that $C_1 \cup C_2 = E(G)$. Let $\vec{G}$ be the *lexicographic orientation* of G induced by $(\vec{C_1}, \vec{C_2})$. That is, we orient each edge $e \in E(G) \cap C_1$ as it is oriented in $\vec{C_1}$, and we orient each $e \in E(G) - C_1$ as it is oriented in $\vec{C_2}$.

Let X be a proper nonempty subset of $V(\vec{G})$. We shall show that $\frac{|\delta^+ X|}{|\delta X|} > \frac{1}{4}$ and the result follows from (1). We associate with every edge $e \in \delta X$ an ordered pair $\sigma\tau \in \{+,-,0\}^2$ where

$$\sigma = \begin{cases} + & \text{if } \vec{C}_1 \text{ traverses } e \text{ from } X \text{ to } V-X \\ - & \text{if } \vec{C}_1 \text{ traverses } e \text{ from } V-X \text{ to } X \\ 0 & \text{if } e \notin C_1 \end{cases}$$

and where τ is defined similarly using C_2 in place of C_1. The pair $\sigma\tau$ is called the *type* of e.

Let $x_{\sigma\tau}$ denote the proportion of edges in δX having type $\sigma\tau$. Since $C_1 \cup C_2 = E(G)$, no edge has type 00. We consider the 8-dimensional column vector

$$x = [x_{++}\ x_{+0}\ x_{+-}\ x_{0+}\ x_{--}\ x_{-0}\ x_{-+}\ x_{0-}]^T.$$

Since each $\vec{C}_i$ traverses δX the same number of times in each direction, x is a feasible point in the polyhedron P of Lemma 6. Because $\vec{G}$ is defined lexicographically, we have

$$\frac{|\delta^+ X|}{|\delta X|} = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]x$$

so this ratio is bounded below by the optimum value of the linear program of Lemma 6. By that lemma, the unique optimum solution is $x^* = \left[\frac{1}{4} 0 0 0 0 0 \frac{1}{4} \frac{1}{2}\right]^T$ with value $z^* = \frac{1}{4}$. Since J_2 is spanning and connected, we have $\delta X - C_2 \neq \emptyset$, so $x_{+0} + x_{-0} > 0$. Thus $x \neq x^*$ and $[1 1 1 1 0 0 0 0]x > \frac{1}{4}$ as claimed. ∎

Remark. We have not proved that $\sup\{\phi_c(G) : G \text{ is 6-edge connected}\} < 4$. To do so by the above method would require finding disjoint V_1-joins J_1, J_2 such that $|J_1 \cap \delta X| \geq c|\delta X|$ for all $X \subseteq V$ and some fixed $c > 0$. For k-edge connected graphs, the strongest lower bound implied directly by Lemmas 3 and 4 is $|J_1 \cap \delta X| \geq \lfloor \frac{k-4}{2} \rfloor$.

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